Periodic solutions for planar autonomous systems with nonsmooth periodic perturbations

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Abstract

In this paper we consider a class of planar autonomous systems having an isolated limit cycle x_0 of smallest period $T > 0$ such that the associated linearized system around it has only one characteristic multiplier with absolute value 1. We consider two functions, defined by means of the eigenfunctions of the adjoint of the linearized system, and we formulate conditions in terms of them in order to have the existence of two geometrically distinct families of $T-$periodic solutions of the autonomous system when it is perturbed by nonsmooth $T-$periodic nonlinear terms of small amplitude. We also show the convergence of these periodic solutions to x_0 as the perturbation disappears and we provide an estimation of the rate of convergence. The employed methods are mainly based on the theory of topological degree and its properties that allow less regularity on the data than that required by the approach, commonly employed in the existing literature on this subject, based on various versions of the implicit function theorem.

Keywords: planar autonomous systems, limit cycles, characteristic multipliers, nonsmooth periodic perturbations, periodic solutions, topological degree.

1. Introduction

Loud in [23] provided conditions under which the perturbed system of ordinary differential equations

$$\dot{x} = \psi(x) + \varepsilon\phi(t, x, \varepsilon), \tag{1}$$

where

$$\psi \in C^2(\mathbb{R}^n, \mathbb{R}^n), \quad \phi \in C^1(\mathbb{R} \times \mathbb{R}^n \times [0, 1], \mathbb{R}^n) \tag{2}$$

and ϕ is T-periodic with respect to time, has, for sufficiently small $\varepsilon > 0$, a T-periodic solution which tends to a T-periodic limit cycle x_0 of the unperturbed system

$$\dot{x} = \psi(x) \tag{3}$$

as $\varepsilon \to 0$. The limit cycle x_0 satisfies the property that the linearized system

$$\dot{y} = \psi'(x_0(t))y \tag{4}$$

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has only one characteristic multiplier with absolute value 1. Here and in the following by $C^i(\mathbb{R}^m, \mathbb{R}^n)$ we denote the vector space of all continuous functions acting from $\mathbb{R}^m$ to $\mathbb{R}^n$ having i-th continuous derivatives. The main tool employed by Loud is the following, so-called bifurcation, function

$$f_0(\theta) = \int_0^T \langle z_0(\tau), \phi(\tau - \theta, x_0(\tau), 0)\rangle \, d\tau, \tag{5}$$

where z_0 is a T-periodic solution of the adjoint system of (4)

$$\dot{z} = -\left(\psi'(x_0(t))\right)^* z, \tag{6}$$

here A^* denotes the transpose of the matrix A. Specifically, Lemma 2 in [23] states that in order that system (1) has a T-periodic solution x_ε such that

$$x_\varepsilon(t - \theta_0) \to x_0(t) \text{ as } \varepsilon \to 0 \tag{7}$$

it is necessary that $\theta_0 \in \mathbb{R}$ be a zero of the equation

$$f_0(\theta) = 0. \tag{8}$$

If (8) is satisfied for some $\theta = \theta_0$ and $f_0'(\theta_0) \neq 0$, i.e. θ_0 is simple, then by ([23], Theorem 1) for all sufficiently small $\varepsilon > 0$ system (1) possesses a T-periodic solution x_ε satisfying

$$\|x_\varepsilon(t - \theta_0) - x_0(t)\| \leq \varepsilon M, \tag{9}$$

where $M > 0$ is a constant. These results are also consequences of general results stated by Malkin in [26].

The function f_0 has been widely employed to treat different problems concerning periodic solutions of system (1) with $\varepsilon > 0$ small. We quote in the sequel some papers from the relevant bibliography devoted to this subject. In [23] Loud also considered the case when (8) is identically satisfied, i.e. $f_0(\theta) = 0$ for any $\theta \in [0, T]$, to treat this case he introduced a new function which plays the role of f_0 and he showed that if this function has a simple zero θ_0 then there exists a family of T-periodic solutions to (1) satisfying (7) (see also [22]). Moreover in [23] it is also considered the case when θ_0 is not a simple zero of f_0, and the problem of the existence of T-periodic solutions to (1) is associated with the problem of the existence of roots of a certain quadratic equation. The case when the limit cycle x_0 of system (3) is not isolated, in particular, when the unperturbed system is Hamiltonian, and the case when the characteristic multiplier of system (4) is not simple have been considered by many authors. If system (3) is not necessary autonomous and it has a multi-parameterized family of T-periodic solutions, then existence of T-periodic solutions of the perturbed system satisfying (15) was proved by Malkin [26]. Melnikov [29] treated the case when the limit cycle is not isolated and the limit cycles near x_0 are of different periods (see also Loud [24] and Kac [14]) and he showed that the simple zeros of suitably defined bifurcation functions $f_{m,n}$, $m, n \in \mathbb{N}$, called Melnikov subharmonic functions, generate periodic solutions in a neighborhood of x_0 whose periods are in $m : n$ ratio with respect to the periods of the perturbation term. Finally, Rhouma and Chicone [34] have considered the case when 1 is not a simple multiplier of the linearized system, to deal with the problem of existence of T-periodic solutions they introduced a new two variables bifurcation function f_0 whose simple zeros determine families of T-periodic solutions satisfying (7).

These theoretical results have been then developed in different directions: Hausrath and Manásevich [10], (see also [11]), found a class of T-periodic perturbations ϕ for which the subharmonic Melnikov function $f_{1,1}$

has at least two simple zeros, obtaining the existence of at least two families of T-periodic solutions to (1) satisfying (7). Makarenkov in [27] provided useful formulas to calculate simple zeros of Malkin's bifurcation function in case when the function ϕ is sinusoidal in time. Tkhai [37] and Lazer [20] developed Malkin's and Melnikov's approaches respectively to study the existence of periodic solutions to (1) satisfying (7) and possessing some additional symmetry properties that represent relevant features in the applications. Farkas in [13] investigated the existence of the so-called D-periodic solutions to (1) which are not necessarily periodic but having periodic derivative. Greenspan and Holmes in [8] and Guckenheimer and Holmes in [7] applied the method of subharmonic Melnikov's functions to a variety of practical problems, a number of applications of Malkin's bifurcation function can be found in the book of Blekhman [1].

In all the previous papers, to show the existence of T-periodic solutions for $\varepsilon > 0$ small, several formulations of the implicit function theorem have been employed. Therefore, condition (2) is the common assumption of these papers (sometimes it is even required more regularity on ψ and ϕ). The persistence of the limit cycle x_0 under less restrictive regularity assumptions than (2) is studied only for the cases when system (3) is linear, in this case the modified averaging methods developed by Mitropol'sksii [30] and Samoylenko [35] can be applied as well as the coincidence degree theory introduced by Mawhin, see, for instance, ([28], Theorem IV.13); Hamiltonian, see M. Henrard and F. Zanolin [12]; or piecewise differentiable, see Kolovskiĭ [17] and Šteĭnberg [36].

In the present paper we assume that the linearized system (4) has only one characteristic multiplier equal to 1 and

$$\psi \in C^1(\mathbb{R}^2, \mathbb{R}^2), \quad \phi \in C(\mathbb{R} \times \mathbb{R}^2 \times [0,1], \mathbb{R}^2). \tag{10}$$

By combining the function f_0 with the analogously defined function

$$f_1(\theta, s) = \int_{s-T}^{s} \langle z_1(\tau), \phi(\tau - \theta, x_0(\tau), 0)\rangle \, d\tau,$$

where z_1 is an eigenfunction of system (6) corresponding to the characteristic multiplier $\rho_* \neq 1$, we give conditions in Theorem 3 for the existence of T-periodic solutions to (1) satisfying (7). Although, as we have mentioned before, in many papers it was proved the existence of two or more families of T-periodic solutions to (1) converging to x_0 in the sense of (7), it was not guaranteed that these families do not coincide geometrically, namely if one is just a shift in time of the other. In this paper our results ensure the existence of at least two geometrically distinct families of T-periodic solutions to (1) satisfying (7). Moreover, since property (9) is a consequence of the application of the implicit function theorem it is not anymore guaranteed under our conditions (10). However, we will show in Theorem 1 that under conditions (10) the following property holds

$$\varepsilon M_1 |f_1(\theta_0, t)| \leq \|x_\varepsilon(t - \theta_\varepsilon(t)) - x_0(t)\| + o(\varepsilon) \leq \varepsilon M_2 |f_1(\theta_0, t)| \quad \text{for any } \varepsilon \in (0, \varepsilon_0) \text{ and any } t \in [0, T], \tag{11}$$

where $0 < M_1 < M_2$ and $\theta_\varepsilon(t) \to \theta_0$ as $\varepsilon \to 0$ uniformly with respect to $t \in [0, T]$. The introduction of the function f_1, as shown by (11) and Corollaries 1 and 2 of this paper, gives a new qualitative information about the convergence (7) with respect to (9) and it is a contribution to the problem posed by Hale and Táboas in [9] concerning the behavior of the periodic solutions of a second order periodically perturbed autonomous system when the perturbation disappears. We would like also to remark, that Loud in [23] provided a precise information about the way of convergence of x_ε to x_0 by means of the representation $x_\varepsilon(t) = x_0(t + \theta_0) + \varepsilon y(t + \theta_0) + o(\varepsilon)$,

where the function y is a suitably chosen solution of system (60) of this paper with $\xi = x_0(0)$, see ([23], formulas 1.3 and 2.11).

In order to prove the existence of T-periodic solutions to (1) satisfying (7) under assumptions (10) we make use of the topological degree theory. Specifically, for $\varepsilon > 0$, we consider the integral operator $G_\varepsilon : C([0,T],\mathbb{R}^2) \to C([0,T],\mathbb{R}^2)$ given by $(G_\varepsilon x)(t) = x(T) + \int_0^t \psi(x(\tau))d\tau + \varepsilon \int_0^t \phi(\tau, x(\tau), \varepsilon)d\tau$, $t \in [0,T]$, here $C([0,T],\mathbb{R}^2)$ is the Banach space of all the continuous functions defined on $[0,T]$ with values in $\mathbb{R}^2$ equipped with the sup-norm. We also consider the Leray-Schauder degree $d(I - G_\varepsilon, W_U)$, see Brown ([4], §9), of the compact vector field $I - G_\varepsilon$ with respect to the open set $W_U = \{x \in C([0,T],\mathbb{R}^2) : x(t) \in U, \text{ for any } t \in [0,T]\}$, where U is an open set of $\mathbb{R}^2$. We will provide conditions in terms of the functions f_0 and f_1 ensuring that $d(I-G_\varepsilon, W_{U_0}) \neq d(I-G_\varepsilon, W_{U_\varepsilon})$ for all $\varepsilon > 0$ sufficiently small, where U_0 is the interior of the limit cycle x_0 and $U_\varepsilon \subset U_0$ (or $U_0 \subset U_\varepsilon$) is a suitably defined family of sets such that $U_\varepsilon \to U_0$ as $\varepsilon \to 0$. To do this we use a result by Capietto, Mawhin and Zanolin ([5], Corollary 1) which, under our assumptions, states that $d(I - G_0, W_{U_\varepsilon}) = 1$ for $\varepsilon > 0$ sufficiently small. Then in Theorem 2, by means of a result due to Kamenskii, Makarenkov and Nistri ([15], Theorem 2), we conclude that there exists a continuous vector field $F : \mathbb{R}^2 \to \mathbb{R}^2$ with

$$F(x_0(\theta)) = f_0(\theta)\dot{x}_0(\theta) + f_1(\theta,\theta)y_1(\theta), \quad \text{for any } \theta \in [0,T], \tag{12}$$

here y_1 denotes the eigenfunction of (4), corresponding to the characteristic multiplier $\rho \neq 1$, such that for all $\varepsilon > 0$ sufficiently small we have that $d(I - G_\varepsilon, W_{U_0}) = d_B(F, U_0)$, where $d_B(F, U_0)$ is the Brouwer degree of F on U_0, see e.g. Brown ([4], §8). In our case the integer $d_B(F, U_0)$ can be easily calculated since it is equal to the Poincaré index of x_0 with respect to the vector field F multiplied by $+1$ or -1 according with the orientation of x_0, see Lefschetz ([21], Ch. IX, §4). Furthermore, as it was observed by Bobylev and Krasnoselskii in [2], for any small neighborhood $B_\delta(\partial W_{U_0})$ of the boundary ∂W_{U_0} of W_{U_0}, we have that $d(I - G_0, B_\delta(\partial W_{U_0})) = 0$ and so one cannot directly apply Leray-Schauder fixed point theorem for studying the existence of T-periodic solutions to (1) satisfying (7).

The paper is organized as follows. Theorem 1 of Section 2 states property (11) for the T-periodic solutions of system (1), in Theorem 2 we prove the coincidence degree formula $d(I - G_\varepsilon, W_{U_0}) = d_B(F, U_0)$ for $\varepsilon > 0$ small. Finally, we give the main result of the paper: Theorem 3 which states the existence of at least two geometrically distinct families of T-periodic solutions to (1) satisfying (7). In Section 3 we provide an example which shows how formula (12) can be used for the practical calculation of $d_B(F, U_0)$. In fact, under quite general conditions on f_0 and f_1, we show that if $\phi(t,\xi) = -\phi(t + T/2, \xi)$ for any $t \in [0,T]$ and $\xi \in \mathbb{R}^2$, then $d_B(F, U_0) \in \{0, 2\}$. Finally, we outline some methods for calculating the eigenfunctions y_1, z_0 and z_1.

2. Main results.

Through the paper we assume the following condition:

$(A_0)-$ system (3) has a limit cycle x_0 with smallest period $T > 0$ and the linearized system (4) has only one characteristic multiplier equal to 1.

In what follows we provide the notations that we will use in the proofs of the results of this Section.

By y_1 we denote the eigenfunction of (4) corresponding to the characteristic multiplier $\rho \neq 1$ (clearly $\dot{x}_0$ is the eigenfunction of (4) corresponding to the characteristic multiplier 1). Moreover, z_0 and z_1 will denote the

eigenfunctions of (6) corresponding to the characteristic multipliers 1 and $\rho_* \neq 1$ respectively. (a_1, a_2) is the matrix whose columns are the vectors $a_1, a_2 \in \mathbb{R}^2$, $(a_1, a_2)^*$ denotes the transpose of (a_1, a_2), $\Omega(\cdot, t_0, \xi)$ is the solution of system (3) satisfying $x(t_0) = \xi$, $\Omega'_\xi(\cdot, t_0, \xi)$ is the derivative of $\Omega(\cdot, t_0, \xi)$ with respect to the third variable, U_0 is the interior of the limit cycle x_0 of system (3), ∂U_0 is the boundary of U_0, $[v]_i$, $i = 1, 2$, is the i-th component of vector $v \in \mathbb{R}^2$. $a \parallel b$ will indicate that the vectors $a, b \in \mathbb{R}^2$ are parallel, $a^\perp$ denotes the vector $a \in \mathbb{R}^2$ rotated of $\pi/2$ clockwise and $\angle(a, b) = \arccos \dfrac{|\langle a, b\rangle|}{\|a\| \cdot \|b\|}$ is the angle between the two vectors $a, b \in \mathbb{R}^2$. By $o(\varepsilon)$, $\varepsilon > 0$, we will denote a function, which may depend also on other variables having the property that $\dfrac{o(\varepsilon)}{\varepsilon} \to 0$ as $\varepsilon \to 0$ uniformly with respect to the other variables when they belong to a bounded set.
Finally, let $t, r \in \mathbb{R}$ and let $h(t, r)$ be the vector of $\mathbb{R}^2$ given by

$$h(t, r) = x_0(t) + r \frac{z_0(t)^\perp}{\|z_0(t)^\perp\|}. \tag{13}$$

Define the function $(t, r) \to I(t, r)$ as follows

$$I(t, r) = \Omega(T, 0, h(t, r)). \tag{14}$$

It is easily seen that, for any $t \in [0, T]$, the curve $r \to I(t, r)$ intersects the limit cycle x_0 at the point $I(t, 0) = x_0(t)$.

The following theorem states a property similar to (9) in the case when the autonomous system (3) is perturbed by nonsmooth functions ϕ.

Theorem 1. *Assume conditions (10). Assume that, for all sufficiently small $\varepsilon > 0$, system (1) has a T-periodic solution x_ε satisfying*

$$x_\varepsilon(t - \theta_0) \to x_0(t) \quad as \quad \varepsilon \to 0, \tag{15}$$

for any $t \in [0, T]$, where $\theta_0 \in [0, T]$. Then there exist constants $0 < M_1 < M_2$, $\varepsilon_0 > 0$ and $r_0 \in (0, 1]$ such that

$$\varepsilon M_1 |f_1(\theta_0, t)| \leq \|x_\varepsilon(t - \theta_\varepsilon(t)) - x_0(t)\| + o(\varepsilon) \leq \varepsilon M_2 |f_1(\theta_0, t)| \quad \textit{for any } \varepsilon \in (0, \varepsilon_0) \textit{ and any } t \in [0, T], \tag{16}$$

where $\theta_\varepsilon(t) \to \theta_0$ as $\varepsilon \to 0$ uniformly with respect to $t \in [0, T]$, and $x_\varepsilon(t - \theta_\varepsilon(t)) \in I(t, [-r_0, r_0])$, $t \in [0, T]$.

To prove Theorem 1 we need some preliminary lemmas.

Lemma 1. *For any $t \in \mathbb{R}$ we have*

$$(\dot{x}_0(t) \ y_1(t))^* (z_0(t) \ z_1(t)) = \begin{pmatrix} \langle \dot{x}_0(0), z_0(0)\rangle & 0 \\ 0 & \langle y_1(0), z_1(0)\rangle \end{pmatrix}. \tag{17}$$

Proof. By Perron's lemma [32] (see also Demidovich ([6], Sec. III, §12) for any $t \in \mathbb{R}$ we have

$$(\dot{x}_0(t) \ y_1(t))^* (z_0(t) \ z_1(t)) := \begin{pmatrix} \langle \dot{x}_0(t), z_0(t)\rangle & \langle \dot{x}_0(t), z_1(t)\rangle \\ \langle y_1(t), z_0(t)\rangle & \langle y_1(t), z_1(t)\rangle \end{pmatrix} = \begin{pmatrix} \langle \dot{x}_0(0), z_0(0)\rangle & \langle \dot{x}_0(0), z_1(0)\rangle \\ \langle y_1(0), z_0(0)\rangle & \langle y_1(0), z_1(0)\rangle \end{pmatrix}.$$

Thus, in particular, $\langle \dot{x}_0(0), z_1(0)\rangle = \langle \dot{x}_0(T), z_1(T)\rangle$. On the other hand $\dot{x}_0(0) = \dot{x}_0(T)$ and $z_1(T) = \rho_* z_1(0)$, $\rho_* \neq 1$, thus $\langle \dot{x}_0(0), z_1(0)\rangle = 0$. Analogously, since $y_1(T) = \rho y_1(0)$, $\rho \neq 1$, and $z_0(0) = z_0(T)$, we have that $\langle y_1(0), z_0(0)\rangle = 0$.

□

Lemma 2. *Under the assumptions of Theorem 1 there exist $r_0 \in (0,1]$ and $\alpha_0 \in [0, \pi/2)$ such that*

$$\angle(I(t,r) - x_0(t), \dot{x}_0(t)^\perp) < \alpha_0 \quad \text{for any } t \in [0,T] \quad \text{and any } r \in [-r_0, r_0]. \tag{18}$$

Proof. Assume the contrary, hence there exist sequences $\{t_n\}_{n\in\mathbb{N}} \subset [0,T]$, $t_n \to t_0$ as $n \to \infty$, $\{r_n\}_{n\in\mathbb{N}} \subset (0,1]$, $r_n \to 0$ as $n \to \infty$ such that

$$\angle(I(t_n, r_n) - x_0(t_n), \dot{x}_0(t_n)^\perp) \to \pi/2 \quad \text{as } n \to \infty. \tag{19}$$

We have

$$I(t_n,r_n) - x_0(t_n) = \Omega\left(T, 0, x_0(t_n) + r_n \frac{z_0(t_n)^\perp}{\|z_0(t_n)^\perp\|}\right) - x_0(t_n) = r_n \Omega'_\xi(T,0,x_0(t_n)) \frac{z_0(t_n)^\perp}{\|z_0(t_n)^\perp\|} + o(r_n). \tag{20}$$

By Theorem 2.1 of [18] it follows that $\Omega'_\xi(T,0,h(t,0)) = Y(T,t)$ where $Y(\cdot,t)$ is the fundamental matrix for the system

$$\dot{y}(\tau) = \psi'(x_0(\tau + t))y(\tau), \tag{21}$$

satisfying $Y(0,t) = I$, thus $\Omega'_\xi(T,0,h(t,0))y_1(t) = \rho y_1(t)$. On the other hand from Lemma 1 we have

$$y_1(t) \parallel z_0(t)^\perp, \tag{22}$$

therefore $\Omega'_\xi(T,0,h(t,0))z_0(t)^\perp = \rho z_0(t)^\perp$ and (20) can be rewritten as follows

$$I(t_n,r_n) - x_0(t_n) = \rho r_n \frac{z_0(t_n)^\perp}{\|z_0(t_n)^\perp\|} + o(r_n). \tag{23}$$

Hence

$$\angle(I(t_n,r_n) - x_0(t_n), \dot{x}_0(t_n)^\perp) = \arccos \frac{\left|\left\langle \rho \dfrac{z_0(t_n)^\perp}{\|z_0(t_n)^\perp\|} + \dfrac{o(r_n)}{r_n}, \dot{x}_0(t_n)^\perp \right\rangle\right|}{\left\|\rho \dfrac{z_0(t_n)^\perp}{\|z_0(t_n)^\perp\|} + \dfrac{o(r_n)}{r_n}\right\| \cdot \|\dot{x}_0(t_n)^\perp\|}.$$

Without loss of generality we may assume

$$\langle \dot{x}_0(0), z_0(0) \rangle = 1 \tag{24}$$

and so

$$\angle(I(t_n,r_n) - x_0(t_n), \dot{x}_0(t_n)^\perp) \to \arccos \frac{1}{\|z_0(t_0)^\perp\| \cdot \|\dot{x}_0(t_0)^\perp\|} \qquad \text{as } n \to \infty$$

contradicting (19). Therefore there exist $r_0 \in (0,1]$ and $\alpha_0 \in [0,\pi/2)$ satisfying (18).

□

We can now prove the following.

Lemma 3. *Under the assumptions of Theorem 1 there exists $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0)$ and any $t \in [0,T]$ we have*

$$x_\varepsilon(t - \theta_\varepsilon(t)) \in I(t, [-r_0, r_0])$$

where $\theta_\varepsilon(t) = \theta_0 - \Delta_\varepsilon(t)$, $\Delta_\varepsilon(t) \in [t - \frac{T}{2}, t + \frac{T}{2}]$ and $\Delta_\varepsilon(t) \to 0$ as $\varepsilon \to 0$ uniformly in $t \in [0,T]$. Moreover, there exists $M > 0$ such that

$$\|x_\varepsilon(t - \theta_\varepsilon(t)) - x_0(t)\| \le \varepsilon M \quad \textit{for any } \varepsilon \in (0,\varepsilon_0) \quad \textit{and any } t \in [0,T]. \tag{25}$$

Proof. First of all observe that $r_0 > 0$ given by Lemma 2 can be chosen to satisfy

$$I(t,[-r_0,r_0]) \cap x_0([0,T]) = \{x_0(t)\}, \text{ for any } t \in [0,T]. \tag{26}$$

From (18) of Lemma 2 and (15) we have that there exists $\varepsilon_0 > 0$ such that $I(t,[-r_0,r_0]) \cap x_\varepsilon([0,T]) \neq \emptyset$ for any $\varepsilon \in (0,\varepsilon_0)$ and any $t \in [0,T]$. Hence, for any $\varepsilon \in (0,\varepsilon_0)$ and $t \in [0,T]$ there exists $\Delta_\varepsilon(t) \in \left[t - \frac{T}{2}, t + \frac{T}{2}\right]$ such that

$$x_\varepsilon(t - \theta_0 + \Delta_\varepsilon(t)) \in I(t,[-r_0,r_0]). \tag{27}$$

We claim that

$$\Delta_\varepsilon(t) \to 0 \text{ as } \varepsilon \to 0 \tag{28}$$

uniformly with respect to $t \in [0,T]$. In fact, assume the contrary, thus there exist sequences $\{\varepsilon_n\}_{n\in\mathbb{N}} \subset (0,\varepsilon_0)$, $\varepsilon_n \to 0$ as $n \to \infty$, and $\{t_n\}_{n\in\mathbb{N}}$, $t_n \to t_0 \in [0,T]$ as $n \to \infty$, such that $\Delta_{\varepsilon_n}(t_n) \to \Delta_0 \neq 0$ and $x_{\varepsilon_n}(t_n - \theta_0 + \Delta_{\varepsilon_n}(t_n)) \in I(t_n,[-r_0,r_0])$. Since from (26) we have $I(t_n,[-r_0,r_0]) \cap x_0([0,T]) = \{x_0(t_n)\}$ then

$$x_{\varepsilon_n}(t_n - \theta_0 + \Delta_{\varepsilon_n}(t_n)) \to x_0(t_0) \quad \text{as } n \to \infty. \tag{29}$$

Applying (15) we have

$$x_{\varepsilon_n}(t_n - \theta_0 + \Delta_{\varepsilon_n}(t_n)) \to x_0(\Delta_0 + t_0). \tag{30}$$

From (29) and (30) we conclude that

$$x_0(t_0) = x_0(\Delta_0 + t_0), \tag{31}$$

where $\Delta_0 \in \left[t_0 - \frac{T}{2}, t_0 + \frac{T}{2}\right]$, since T is the smallest period of x_0 it follows from (31) that $\Delta_0 = 0$, which is a contradiction.

Pick any $\tau \in [0,T]$, in what follows we show that the shifts $t \to \Delta_\varepsilon(t)$ have the property that the convergence of $x_\varepsilon(\tau + t - \theta_\varepsilon(t))$ to $x_0(\tau + t)$ is of order $\varepsilon > 0$, where $\theta_\varepsilon(t) = \theta_0 - \Delta_\varepsilon(t)$, and thus the claim of Lemma 3 is proved.

For this consider the change of variables $\nu_\varepsilon(\tau,t) = \Omega(0,\tau,x_\varepsilon(\tau + t - \theta_\varepsilon(t)))$ in system (1). It is clear that $x_\varepsilon(\tau + t - \theta_\varepsilon(t)) = \Omega(\tau,0,\nu_\varepsilon(\tau,t))$ and so

$$\dot{x}_\varepsilon(\tau + t - \theta_\varepsilon(t)) = \psi(\Omega(\tau,0,\nu_\varepsilon(\tau,t)) + \Omega'_\xi(\tau,0,\nu_\varepsilon(\tau,t))(\nu_\varepsilon)'_\tau(\tau,t). \tag{32}$$

On the other hand from (1) we have

$$\dot{x}_\varepsilon(\tau + t - \theta_\varepsilon(t)) = \psi(\Omega(\tau,0,\nu_\varepsilon(\tau,t))) + \varepsilon\phi(\tau + t - \theta_\varepsilon(t),\Omega(\tau,0,\nu_\varepsilon(\tau,t)),\varepsilon). \tag{33}$$

From (32) and (33) it follows

$$(\nu_\varepsilon)'_\tau(\tau,t) = \varepsilon \left(\Omega'_\xi(\tau,0,\nu_\varepsilon(\tau,t))\right)^{-1} \phi(\tau + t - \theta_\varepsilon(t),\Omega(\tau,0,\nu_\varepsilon(\tau,t)),\varepsilon)$$

and since

$$\nu_\varepsilon(0,t) = x_\varepsilon(t - \theta_\varepsilon(t)) = x_\varepsilon(T + t - \theta_\varepsilon(t)) = \Omega(T,0,\nu_\varepsilon(T,t)) \tag{34}$$

we finally obtain

$$\nu_\varepsilon(\tau,t) = \Omega(T,0,\nu_\varepsilon(T,t)) + \varepsilon \int_0^\tau \left(\Omega'_\xi(s,0,\nu_\varepsilon(s,t))\right)^{-1} \phi(s + t - \theta_\varepsilon(t),\Omega(s,0,\nu_\varepsilon(s,t)),\varepsilon)ds. \tag{35}$$

Since $\nu_\varepsilon(\tau,t) \to \Omega(0,\tau,x_0(\tau+t)) = x_0(t)$ as $\varepsilon \to 0$ we can write $\nu_\varepsilon(\tau,t)$ in the following form

$$\nu_\varepsilon(\tau,t) = x_0(t) + \varepsilon\mu_\varepsilon(\tau,t). \tag{36}$$

Subtract $x_0(t)$ from both sides of (35) obtaining

$$\begin{aligned}\varepsilon\mu_\varepsilon(\tau,t) &= \varepsilon\,\Omega'_\xi(T,0,x_0(t))\mu_\varepsilon(T,t) + o(\varepsilon\mu_\varepsilon(T,t)) + \\ &+ \varepsilon \int_0^\tau \left(\Omega'_\xi(s,0,\nu_\varepsilon(s,t))\right)^{-1} \phi(s+t-\theta_\varepsilon(t),\Omega(s,0,\nu_\varepsilon(s,t)),\varepsilon)ds.\end{aligned} \tag{37}$$

Since $x_\varepsilon(t-\theta_\varepsilon(t)) \in I(t,[-r_0,r_0])$ then from (14) there exists $r_\varepsilon(t) \in [-r_0,r_0]$ such that $x_\varepsilon(t-\theta_\varepsilon(t)) = \Omega(T,0,h(t,r_\varepsilon(t)))$ and by (13) we get

$$\begin{aligned}\varepsilon\mu_\varepsilon(T,t) &= \nu_\varepsilon(T,t) - x_0(t) = \Omega(0,T,x_\varepsilon(t-\theta_\varepsilon(t))) - x_0(t) = \\ &= \Omega(0,T,\Omega(T,0,h(t,r_\varepsilon(t)))) - x_0(t) = h(t,r_\varepsilon(t)) - x_0(t) = r_\varepsilon(t)\frac{z_0(t)^\perp}{\|z_0(t)^\perp\|}.\end{aligned}$$

Therefore $\mu_\varepsilon(T,t) \parallel z_0(t)^\perp$ and by (22) we can rewrite (37) as follows

$$\varepsilon\mu_\varepsilon(\tau,t) = \varepsilon\rho\mu_\varepsilon(T,t) + o(\varepsilon\mu_\varepsilon(T,t)) + \varepsilon\int_0^\tau \left(\Omega'_\xi(s,0,\nu_\varepsilon(s,t))\right)^{-1}\phi(s+t-\theta_\varepsilon(t),\Omega(s,0,\nu_\varepsilon(s,t)),\varepsilon)ds. \tag{38}$$

We now prove that the functions $(\tau,t) \to \mu_\varepsilon(\tau,t)$ are uniformly bounded with respect to $\varepsilon \in (0,\varepsilon_0)$. For this we argue by contradiction, therefore there exist sequences $\{\varepsilon_n\}_{n\in\mathbb{N}} \subset (0,1)$, $\varepsilon_n \to 0$ as $n \to \infty$, $\{\tau_n\}_{n\in\mathbb{N}} \subset [0,T]$, $\tau_n \to \tau_0$ as $n \to \infty$ and $\{t_n\}_{n\in\mathbb{N}} \subset [0,T]$, $t_n \to t_0$ as $n \to \infty$, such that $\|\mu_{\varepsilon_n}(\tau_n,t_n)\| \to \infty$ as $n \to \infty$, so $\|\mu_{\varepsilon_n}(\cdot,t_n)\|_C \to \infty$ as $n \to \infty$, where $\|\cdot\|_C$ is the usual sup-norm of $C([0,T],\mathbb{R}^2)$. Let $q_n(\tau) = \dfrac{\mu_{\varepsilon_n}(\tau,t_n)}{\|\mu_{\varepsilon_n}(\cdot,t_n)\|_C}$, then from (38) we have

$$\begin{aligned}q_n(\tau) &= \rho q_n(T) + \frac{o(\varepsilon_n\mu_{\varepsilon_n}(T,t_n))}{\varepsilon_n\|\mu_{\varepsilon_n}(\cdot,t_n)\|_C} + \\ &+ \frac{1}{\|\mu_{\varepsilon_n}(\cdot,t_n)\|_C}\int_0^\tau \left(\Omega'_\xi(s,0,\nu_{\varepsilon_n}(s,t_n))\right)^{-1}\phi(s+t_n-\theta_{\varepsilon_n}(t_n),\Omega(s,0,\nu_{\varepsilon_n}(s,t_n)),\varepsilon_n)ds.\end{aligned} \tag{39}$$

By definition the set of continuous functions $A = \{q_n,\ n \in \mathbb{N}\}$, is bounded and, as it is easy to see from (39), A is also equicontinuous. Therefore, by the Ascoli-Arzela Theorem, see e.g. ([4], Theorem 2.3), we may assume without loss of generality that the sequence $\{q_n\}_{n\in\mathbb{N}}$ is converging. Let $q_0 = \lim_{n\to\infty} q_n$, from (39) we may conclude that

$$q_0(\tau) = \rho q_0(T). \tag{40}$$

By (40) it follows that q_0 is a constant function, thus being $\rho \neq 1$ we have $q_0 = 0$. On the other hand, by the definition of q_n, we have that $\|q_0\|_C = 1$. This contradiction shows the uniform boundedness of the functions μ_ε with respect to $\varepsilon \in (0,\varepsilon_0)$. On the other hand from (34) and (36)we have that

$$x_\varepsilon(t-\theta_\varepsilon(t)) - x_0(t) = \varepsilon\mu_\varepsilon(0,t), \tag{41}$$

and thus the proof is complete.

□

Proof of Theorem 1. We have to prove (16) with $t \to \theta_\varepsilon(t)$ as given in Lemma 3. For this, by Lemma 1, we can represent $x_\varepsilon(\tau+t-\theta_\varepsilon(t)) - x_0(\tau+t)$ as follows

$$x_\varepsilon(\tau+t-\theta_\varepsilon(t)) - x_0(\tau+t) = \varepsilon a_\varepsilon(\tau,t)\dot{x}_0(\tau+t) + \varepsilon b_\varepsilon(\tau,t)y_1(\tau+t), \tag{42}$$

where

$$\varepsilon a_\varepsilon(\tau,t) = \langle z_0(\tau+t), x_\varepsilon(\tau+t-\theta_\varepsilon(t)) - x_0(\tau+t)\rangle \qquad \text{and}$$
$$\varepsilon b_\varepsilon(\tau,t) = \langle z_1(\tau+t), x_\varepsilon(\tau+t-\theta_\varepsilon(t)) - x_0(\tau+t)\rangle\,. \tag{43}$$

By Lemma 1 we have that $\langle \dot{x}_0(t), z_1(t)\rangle = 0$, for any $t\in[0,T]$, and so $\dot{x}_0(t)^\perp = k\dfrac{\|\dot{x}_0(t)^\perp\|}{\|z_1(t)\|}z_1(t)$, where $k=+1$ or $k=-1$. Therefore

$$\left\langle \dot{x}_0(\tau+t)^\perp, x_\varepsilon(\tau+t-\theta_\varepsilon(t)) - x_0(\tau+t)\right\rangle = \varepsilon b_\varepsilon(\tau,t)k\frac{\|\dot{x}_0(\tau+t)^\perp\|}{\|z_1(\tau+t)\|}. \tag{44}$$

We aim now at providing an explicit form for (44) by looking for a suitable formula for the function $(\tau,t)\to b_\varepsilon(\tau,t)$. To do this we substract (3) where $x(\tau)$ is replaced by $x_0(\tau+t)$ from (1) where $x(\tau)$ is replaced by $x_\varepsilon(\tau+t-\theta_\varepsilon(t))$ to obtain

$$\begin{aligned}&\dot{x}_\varepsilon(\tau+t-\theta_\varepsilon(t)) - \dot{x}_0(\tau+t) = \psi'(x_0(\tau+t))(x_\varepsilon(\tau+t-\theta_\varepsilon(t)) - x_0(\tau+t)) + \\ &+\varepsilon\phi(\tau+t-\theta_\varepsilon(t), x_\varepsilon(\tau+t-\theta_\varepsilon(t)),\varepsilon) + o(x_\varepsilon(\tau+t-\theta_\varepsilon(t)) - x_0(\tau+t)).\end{aligned} \tag{45}$$

By substituting (42) into (45) and taking into account that

$$\varepsilon a_\varepsilon(\tau,t)\psi'(x_0(\tau+t))\dot{x}_0(\tau+t) = \varepsilon a_\varepsilon(\tau,t)\ddot{x}_0(\tau+t) \qquad \text{and}$$
$$\varepsilon b_\varepsilon(\tau,t)\psi'(x_0(\tau+t))y_1(\tau+t) = \varepsilon b_\varepsilon(\tau,t)\dot{y}_1(\tau+t)$$

we have

$$\begin{aligned}\varepsilon\dot{x}_0(\tau+t)(a_\varepsilon)'_\tau(\tau,t) + \varepsilon y_1(\tau+t)(b_\varepsilon)'_\tau(\tau,t) \quad &= \quad \varepsilon\phi(\tau+t-\theta_\varepsilon(t), x_\varepsilon(\tau+t-\theta_\varepsilon(t))) + \\ &\quad + o(x_\varepsilon(\tau+t-\theta_\varepsilon(t)) - x_0(\tau+t)),\end{aligned}$$

and so

$$\begin{aligned}\varepsilon(b_\varepsilon)'_\tau(\tau,t) \quad &= \quad \varepsilon\left\langle z_1(\tau+t), \phi(\tau+t-\theta_\varepsilon(t), x_\varepsilon(\tau+t-\theta_\varepsilon(t)))\right\rangle + \\ &\quad + \left\langle z_1(\tau+t), o(x_\varepsilon(\tau+t-\theta_\varepsilon(t)) - x_0(\tau+t))\right\rangle.\end{aligned} \tag{46}$$

Moreover, since $z_1(\tau) = \rho_* z_1(\tau-T)$, from (43) it follows that

$$b_\varepsilon(\tau,t) = \rho_* b_\varepsilon(\tau-T,t). \tag{47}$$

System (46)-(47) has a unique solution which, as it is easy to verify, is given by the formula

$$\begin{aligned}b_\varepsilon(\tau,t) \quad &= \quad \frac{\rho_*}{\rho_*-1}\int_{\tau-T}^{\tau}\langle z_1(s+t), \phi(s+t-\theta_\varepsilon(t), x_\varepsilon(s+t-\theta_\varepsilon(t)),\varepsilon)\rangle\, ds + \\ &\quad + \frac{\rho_*}{\rho_*-1}\int_{\tau-T}^{\tau}\left\langle z_1(s+t), \frac{o(x_\varepsilon(s+t-\theta_\varepsilon(t)) - x_0(s+t))}{\varepsilon}\right\rangle ds.\end{aligned}$$

By substituting this formula into (44) we obtain

$$\begin{aligned}&\left\langle \dot{x}_0(\tau+t)^\perp, x_\varepsilon(\tau+t-\theta_\varepsilon(t)) - x_0(\tau+t)\right\rangle = \\ &= \varepsilon k\frac{\|\dot{x}_0(\tau+t)^\perp\|\rho_*}{\|z_1(\tau+t)\|(\rho_*-1)}\int_{\tau-T}^{\tau}\langle z_1(s+t), \phi(s+t-\theta_\varepsilon(t), x_\varepsilon(s+t-\theta_\varepsilon(t)),\varepsilon)\rangle\, ds + \\ &+\varepsilon k\frac{\|\dot{x}_0(\tau+t)^\perp\|\rho_*}{\|z_1(\tau+t)\|(\rho_*-1)}\int_{\tau-T}^{\tau}\left\langle z_1(s+t), \frac{o(x_\varepsilon(s+t-\theta_\varepsilon(t)) - x_0(s+t))}{\varepsilon}\right\rangle ds,\end{aligned} \tag{48}$$

where $\dfrac{o(x_\varepsilon(s+t-\theta_\varepsilon(t))-x_0(s+t))}{\varepsilon}\to 0$ as $\varepsilon\to 0$ uniformly in $s\in[-T,T]$ in virtue of (25). On the other hand

$$\begin{aligned}&\left\langle \dot{x}_0(\tau+t)^\perp, x_\varepsilon(\tau+t-\theta_\varepsilon(t))-x_0(\tau+t)\right\rangle=\\ =\;&\left\|\dot{x}_0(\tau+t)^\perp\right\|\,\|x_\varepsilon(\tau+t-\theta_\varepsilon(t))-x_0(\tau+t)\|\,\cos\angle\left(\dot{x}_0(\tau+t)^\perp, x_\varepsilon(\tau+t-\theta_\varepsilon(t))-x_0(\tau+t)\right)\end{aligned}$$

and by taking into account (18) of Lemma 2, (48) and the fact that $\|\dot{x}_0(t)^\perp\|\neq 0$ for any $t\in[0,T]$ we get

$$\begin{aligned}\|x_\varepsilon(t-\theta_\varepsilon(t))-x_0(t)\| &= \varepsilon g(t)\int_{-T}^{0}\langle z_1(s+t),\phi(s+t-\theta_\varepsilon(t),x_\varepsilon(s+t-\theta_\varepsilon(t)),\varepsilon)\rangle\,ds+\\ &\quad+\varepsilon g(t)\int_{-T}^{0}\left\langle z_1(s+t),\frac{o(x_\varepsilon(s+t-\theta_\varepsilon(t))-x_0(s+t))}{\varepsilon}\right\rangle ds,\end{aligned}$$

where

$$g(t)=\frac{k\rho_*}{\|z_1(t)\|(\rho_*-1)\,\cos\angle\left(\dot{x}_0(t)^\perp,x_\varepsilon(t-\theta_\varepsilon(t))-x_0(t)\right)}$$

is a continuous function on $[0,T]$ with $g(t)\neq 0$ for any $t\in[0,T]$, Therefore,

$$\|x_\varepsilon(t-\theta_\varepsilon(t))-x_0(t)\|=\varepsilon g(t)\int_{-T}^{0}\langle z_1(s+t),\phi(s+t-\theta_\varepsilon(t),x_\varepsilon(s+t-\theta_\varepsilon(t)),\varepsilon)\rangle\,ds+o(\varepsilon). \tag{49}$$

On the other hand from Lemma 3 we have that $\Delta_\varepsilon(t)\to 0$ uniformly in $t\in[0,T]$, thus we can rewrite (49) as follows

$$\|x_\varepsilon(t-\theta_\varepsilon(t))-x_0(t)\|=\varepsilon g(t)\int_{-T}^{0}\langle z_1(s+t),\phi(s+t-\theta_0,x_0(s+t),0)\rangle\,ds+o(\varepsilon),$$

introducing the change of variable $s+t=u$ in the integral we finally get

$$\|x_\varepsilon(t-\theta_\varepsilon(t))-x_0(t)\|=\varepsilon g(t)\int_{t-T}^{t}\langle z_1(u),\phi(u-\theta_0,x_\varepsilon(u-\theta_0),0)\rangle\,du+o(\varepsilon)$$

from which (16) can be directly derived recalling that, by Lemma 3, $x_\varepsilon(t-\theta_\varepsilon(t))\in I(t,[-r_0,r_0])$ for any $\varepsilon\in(0,\varepsilon_0)$ and any $t\in[0,T]$.

□

As a straightforward consequence of Theorem 1 we have the following result.

Corollary 1. *Assume all the conditions of Theorem 1, then for every $t\in[0,T]$ such that*

$$f_1(\theta_0,t)=0$$

we have

$$\|x_\varepsilon(t-\theta_\varepsilon(t))-x_0(t)\|=o(\varepsilon),$$

where $\theta_\varepsilon(t)\to\theta_0$ as $\varepsilon\to 0$ and $x_\varepsilon(t-\theta_\varepsilon(t))\in I(t,[-r_0,r_0])$.

Next result is also a consequence of Theorem 1.

Corollary 2. *Assume all the conditions of Theorem 1. Moreover, assume that*

$$f_1(\theta_0,t)\neq 0\quad \textit{for any}\quad t\in[0,T]. \tag{50}$$

Then there exists $\varepsilon_1>0$ such that $x_\varepsilon(s)\neq x_0(t)$ for any $s,t\in[0,T]$ and any $\varepsilon\in(0,\varepsilon_1)$.

Proof. Let $\varepsilon_0 > 0$ given by Theorem 1. From (50) we can choose $\varepsilon_1 \in (0, \varepsilon_0)$ in such a way that, for any $\varepsilon \in (0, \varepsilon_1)$, we have both

$$o(\varepsilon) < \varepsilon M_1 |f_1(\theta_0, t)|, \quad \text{for any } t \in [0, T], \tag{51}$$

and the validity of (16). Moreover, ε_1 can be also chosen in such a way that there exists $\delta_0 > 0$ such that the curve $\tau \to x_\varepsilon(\tau)$ intersects $I(t, [-\delta_0, \delta_0])$ at only one point for any $\varepsilon \in (0, \varepsilon_1)$ and $t \in [0, T]$. Such a choice is possible, in fact, since $\dot{x}_\varepsilon(\tau - \theta_0) \to \dot{x}_0(\tau)$ as $\varepsilon \to 0$ uniformly with respect to $\tau \in [0, T]$ and the curve $r \to I(t, r)$ intersects the limit cycle x_0 transversally at $r = 0$ for any $t \in [0, T]$, then there exists $\delta_0 > 0$ such that $x_\varepsilon([t - \theta_0 - \delta_0, t - \theta_0 + \delta_0])$ and $I(t, [-\delta_0, \delta_0])$ have only one common point for any $t \in [0, T]$ and sufficiently small $\varepsilon > 0$. On the other hand $\tau \to x_\varepsilon(\tau)$ cannot intersect $I(t, [-\delta_0, \delta_0])$ for $\tau \in \left[t - \theta_0 - \frac{T}{2}, t - \theta_0 - \delta_0\right] \cup \left[t - \theta_0 + \delta_0, t - \theta_0 + \frac{T}{2}\right]$ and $\varepsilon > 0$ sufficiently small, otherwise there would exist sequences $\{\varepsilon_n\}_{n \in \mathbb{N}}$, $\varepsilon_n \to 0$ as $n \to \infty$, $\{t_n\}_{n \in \mathbb{N}}$, $t_n \to t_0 \in [0, T]$ as $n \to \infty$, $\{\tau_n\}_{n \in \mathbb{N}}$, $\tau_n \in [t_n - \theta_0 - \frac{T}{2}, t_n - \theta_0 - \delta_0] \cup [t_n - \theta_0 + \delta_0, t_n - \theta_0 + \frac{T}{2}]$, $\tau_n \to \tau_0 \in \left[t_0 - \theta_0 - \frac{T}{2}, t_0 - \theta_0 - \delta_0\right] \cup \left[t_0 - \theta_0 + \delta_0, t_0 - \theta_0 + \frac{T}{2}\right]$ as $n \to \infty$ such that $x_{\varepsilon_n}(\tau_n) \in I(t_n, [-\delta_0, \delta_0])$, thus $x_0(\tau_0 + \theta_0) = x_0(t_0)$, with $\tau_0 + \theta_0 \neq t_0$ and $|\tau_0 + \theta_0 - t_0| < T$, which contradicts the fact that $T > 0$ is the smallest period of x_0.

To conclude the proof assume now, by contradiction, that there exist $\tilde{\varepsilon} \in (0, \varepsilon_1)$ and $\tilde{s}, \tilde{t} \in [0, T]$ such that $x_{\tilde{\varepsilon}}(\tilde{s}) = x_0(\tilde{t})$. Since $\tau \to x_{\tilde{\varepsilon}}(\tau)$ intersects $I(\tilde{t}, [-\delta_0, \delta_0])$ at only one point then Theorem 1 implies that $\tilde{s} = \tilde{t} - \theta_{\tilde{\varepsilon}}(\tilde{t})$. In conclusion, from (16) we have $\tilde{\varepsilon} M_1 |f_1(\theta_0, \tilde{t})| \leq o(\tilde{\varepsilon})$ contradicting (51).

□

The following result is crucial for the proof of our existence result Theorem 3, but it can be also considered as an independent contribution to the coincidence degree theory.

Theorem 2. *Assume conditions (10). For $\varepsilon > 0$, let $G_\varepsilon : C([0, T], \mathbb{R}^2) \to C([0, T], \mathbb{R}^2)$ be the operator defined by*

$$(G_\varepsilon x)(t) = x(T) + \int_0^t \psi(x(\tau)) d\tau + \varepsilon \int_0^t \phi(\tau, x(\tau), \varepsilon)) d\tau, \qquad t \in [0, T].$$

Let $W_{U_0} = \left\{x \in C([0, T], \mathbb{R}^2) : x(t) \in U_0, \text{ for any } t \in [0, T]\right\}$. Assume that

$$\langle \dot{x}_0(0), z_0(0) \rangle = \langle y_1(0), z_1(0) \rangle = 1. \tag{52}$$

Finally, assume that for every $\theta_0 \in [0, T]$ such that $f_0(\theta_0) = 0$ we have

$$f_1(\theta_0, s + \theta_0) \neq 0, \qquad for\ any\ s \in [0, T]. \tag{53}$$

Then, for all $\varepsilon > 0$ sufficiently small, $I - G_\varepsilon : C([0, T], \mathbb{R}^2) \to C([0, T], \mathbb{R}^2)$ is not degenerate on the boundary of W_{U_0}. Furthermore, there exists a continuous vector field $F : \mathbb{R}^2 \to \mathbb{R}^2$ such that

$$d(I - G_\varepsilon, W_{U_0}) = d_B(F, U_0),$$

where $F(x_0(\theta)) = f_0(\theta)\dot{x}_0(\theta) + f_1(\theta, \theta) y_1(\theta)$ for any $\theta \in [0, T]$.

Some remarks are in order.

Remark 1. *As already observed condition (52) does not affect the generality of Theorem 2.*

Remark 2. *In Theorem 2 we could replace $d_B(F, U_0)$ by $k \cdot \mathrm{ind}(x_0, F)$, where $k = +1$ or $k = -1$ according with the orientation of the limit cycle x_0. Precisely, $k = +1$ if the set U_0 is on the left side when one follows*

∂U_0 according to the parameterization $x_0(t)$ with t increasing from 0 to T, and $k = -1$ in the opposite case. Moreover, $\mathrm{ind}(x_0, F)$ *is the Poincaré index of the trajectory x_0 with respect to the vector field F, namely the total variation of an angle function of the vector $F(x_0(t))$ when t increases from 0 to T, see Lefschetz ([21], Ch. IX, § 4).*

Remark 3. *The Jordan theorem, see Lefschetz ([21], Theorem 4.7), ensures that the interior U_0 of x_0 does exist and it is an open set.*

To prove Theorem 2 we need the following preliminary lemma.

Lemma 4. *For any $s \in [0, T]$, let*

$$F_s(\xi) = \int_{s-T}^{s} \Omega'_\xi(0, \tau, \Omega(\tau, 0, \xi))\phi(\tau, \Omega(\tau, 0, \xi), 0)d\tau, \quad \textit{for any} \quad \xi \in \mathbb{R}^2. \tag{54}$$

Then

$$\begin{aligned} \langle z_0(\theta), F_s(x_0(\theta))\rangle &= f_0(\theta) \quad \textit{for any} \quad s, \theta \in [0, T], \\ \langle z_1(\theta), F_s(x_0(\theta))\rangle &= f_1(\theta, s+\theta) \quad \textit{for any} \quad s, \theta \in [0, T]. \end{aligned} \tag{55}$$

In particular,

$$F_s(x_0(\theta)) = \frac{1}{\langle \dot{x}_0(t), z_0(t)\rangle} f_0(\theta)\dot{x}_0(\theta) + \frac{1}{\langle y_1(t), z_1(t)\rangle} f_1(\theta, s+\theta)y_1(\theta) \quad \textit{for any } s, \theta, t \in [0, T]. \tag{56}$$

Proof. It can be shown, see Krasnosel'skii ([18], Theorem 2.1), that $\Omega'_\xi(t, 0, x_0(\theta)) = Y(t, \theta)$, where $Y(t, \theta)$ is the fundamental matrix of the system

$$\dot{y}(t) = \psi'(x_0(t+\theta))y(t) \tag{57}$$

satisfying $Y(0, \theta) = I$ and since $\Omega'_\xi(0, t, \Omega(t, 0, x_0(\theta))) \cdot \Omega'_\xi(t, 0, x_0(\theta)) = I$ we have

$$F_s(x_0(\theta)) = \int_{s-T}^{s} Y^{-1}(\tau, \theta)\phi(\tau, x_0(\tau+\theta), 0)d\tau. \tag{58}$$

Let us now show that

$$Y^{-1}(t, \theta) = Y(\theta, 0)Y^{-1}(t+\theta, 0). \tag{59}$$

In fact, it is easy to see that $Y(t+\theta, 0)$ is a fundamental matrix for system (57) and so $Y(t+\theta, 0)Y^{-1}(\theta, 0)$ is also a fundamental matrix for (57), moreover we have that $Y(t+\theta, 0)Y^{-1}(\theta, 0) = I$ at $t = 0$. Therefore $Y(t+\theta, 0)Y^{-1}(\theta, 0) = Y(t, \theta)$ which is equivalent to (59).

By substituting (59) into (58) and by the change of variable $\tau + \theta = t$ in the integral of (58) we obtain

$$F_s(x_0(\theta)) = Y(\theta, 0)\int_{s-T}^{s} Y^{-1}(\tau+\theta, 0)\phi(\tau, x_0(\tau+\theta), 0)d\tau = Y(\theta, 0)\int_{s-T+\theta}^{s+\theta} Y^{-1}(t, 0)\phi(t-\theta, x_0(t), 0)dt.$$

Let $Z(t)$ be the fundamental matrix of system (6) given by $Z(t) = Z_0(t)Z_0^{-1}(0)$, where $Z_0(t) = (z_0(t)\ z_1(t))$, $t \in [0, T]$. Since $Y^{-1}(t, 0) = Z^*(t)$, see Perron [32] and Demidovich ([6], Sec. III, §12), then we have

$$F_s(x_0(\theta)) = Y(\theta, 0)\int_{s-T+\theta}^{s+\theta} Y^{-1}(\tau, 0)\phi(\tau-\theta, x_0(\tau), 0)d\tau = (Z_0^*(\theta))^{-1}\int_{s-T+\theta}^{s+\theta} Z_0^*(\tau)\phi(\tau-\theta, x_0(\tau), 0)d\tau.$$

Let

$$\Delta(s, \theta) = \int_{s+\theta-T}^{s+\theta} Z_0^*(\tau)\phi(\tau-\theta, x_0(\tau), 0)d\tau,$$

we have

$$\langle z_i(\theta), F_s(x_0(\theta))\rangle = \left\langle \begin{pmatrix} [z_i(\theta)]_1 \\ [z_i(\theta)]_2 \end{pmatrix}, \begin{pmatrix} [z_0(\theta)]_1 & [z_0(\theta)]_2 \\ [z_1(\theta)]_1 & [z_1(\theta)]_2 \end{pmatrix}^{-1} \Delta(s,\theta) \right\rangle =$$

$$= \left\langle \begin{pmatrix} [z_i(\theta)]_1 \\ [z_i(\theta)]_2 \end{pmatrix}, \frac{1}{\det Z_0(\theta)} \begin{pmatrix} [z_1(\theta)]_2 & -[z_0(\theta)]_2 \\ -[z_1(\theta)]_1 & [z_0(\theta)]_1 \end{pmatrix} \Delta(s,\theta) \right\rangle =$$

$$= \frac{1}{\det Z_0(\theta)} \left\langle \begin{pmatrix} [z_i(\theta)]_1 \\ [z_i(\theta)]_2 \end{pmatrix}, \begin{pmatrix} [z_1(\theta)]_2[\Delta(s,\theta)]_1 - [z_0(\theta)]_2[\Delta(s,\theta)]_2 \\ -[z_1(\theta)]_1[\Delta(s,\theta)]_1 + [z_0(\theta)]_1[\Delta(s,\theta)]_2 \end{pmatrix} \right\rangle =$$

$$= \frac{1}{\det Z_0(\theta)} \{[z_i(\theta)]_1[z_1(\theta)]_2[\Delta(s,\theta)]_1 - [z_i(\theta)]_1[z_0(\theta)]_2[\Delta(s,\theta)]_2 -$$
$$- [z_i(\theta)]_2[z_1(\theta)]_1[\Delta(s,\theta)]_1 + [z_i(\theta)]_2[z_0(\theta)]_1[\Delta(s,\theta)]_2\} =$$

$$= \frac{1}{\det Z_0(\theta)} \{[z_0(\theta)]_1[z_1(\theta)]_2 - [z_0(\theta)]_2[z_1(\theta)]_1\} [\Delta(s,\theta)]_{i+1} =$$

$$= \frac{1}{\det Z_0(\theta)} \det Z_0(\theta) \int_{s+\theta-T}^{s+\theta} \langle z_i(\tau), \phi(\tau-\theta, x_0(\tau), 0)\rangle \, d\tau.$$

For $i = 0, 1$, we obtain (55). Furthermore, from Lemma 1 and (52) we have that

$$F_s(x_0(\theta)) = \frac{1}{\langle \dot{x}_0(t), z_0(t)\rangle} f_0(\theta)\dot{x}_0(\theta) + \frac{1}{\langle y_1(t), z_1(t)\rangle} f_1(\theta, s+\theta) y_1(\theta) \quad \text{for any } \theta \in [0,T] \text{ and any } t \in [0,T].$$

□

Proof of Theorem 2.

Let $\eta(t, s, \xi)$ be the solution of the system

$$\dot{q}(t) = \psi'(\Omega(t,0,\xi))q(t) + \phi(t, \Omega(t,0,\xi)) \tag{60}$$

satisfying $\eta(s, s, \xi) = 0$ whenever $\xi \in \mathbb{R}^2$. It can be shown, see ([16], Lemma 2), that

$$F_s(\xi) = \eta(T, s, \xi) - \eta(0, s, \xi). \tag{61}$$

Therefore, from (52), (53) and (56) we have that

$$\eta(T, s, \xi) - \eta(0, s, \xi) \neq 0 \quad \text{for any } \ \xi \in \partial U_0 \quad \text{and any } \ s \in [0,T], \tag{62}$$

and by applying ([15], Theorem 2) we obtain the existence of an $\varepsilon_0 > 0$ such that

$$d(I - G_\varepsilon, \widetilde{W}_{U_0}) = d_B(\eta(T, 0, \cdot), U_0) \ \text{ for any } \ \varepsilon \in (0, \varepsilon_0),$$

where $\widetilde{W}_{U_0} = \{x \in \mathrm{C}([0,T], \mathbb{R}^2) : \ \Omega(0, t, x(t)) \in U_0 \text{ for any } t \in [0,T]\}$. Since as it is easy to see $\widetilde{W}_{U_0} = W_{U_0}$, by taking into account (56) and (61) we end the proof by defining $F(\xi) = F_0(\xi)$, $\xi \in \mathbb{R}^2$.

□

The following existence theorem is the main result of the paper.

Theorem 3. *Assume (10). Assume that for every zero $\theta_0 \in [0,T]$ of the bifurcation function f_0 we have*

$$f_1(\theta_0, t) \neq 0 \ \ \textit{for any} \ \ t \in [0,T]. \tag{63}$$

Let $F \in C(\mathbb{R}^2, \mathbb{R}^2)$ be a vector field such that on the boundary of U_0 it has the form $F(x_0(\theta)) = f_0(\theta)\dot{x}_0(\theta) + f_1(\theta, \theta)y_1(\theta)$ for any $\theta \in [0,T]$. Assume

$$d_B(F, U_0) \neq 1. \tag{64}$$

Then there exists $\varepsilon_0 > 0$ such that for every $\varepsilon \in (0, \varepsilon_0)$ system (1) has at least two T-periodic solutions $x_{1,\varepsilon}$ and $x_{2,\varepsilon}$ satisfying

$$x_{i,\varepsilon}(t - \theta_i) \to x_0(t) \quad \text{as } \varepsilon \to 0, \ i = 1, 2, \tag{65}$$

where $\theta_1, \theta_2 \in [0, T]$. Moreover, we have that $x_{1,\varepsilon}(t) \in U_0$ and $x_{2,\varepsilon}(t) \notin U_0$, for any $t \in [0, T]$ and any $\varepsilon \in (0, \varepsilon_0)$.

Proof. Denote by $W_\delta(\partial U_0)$ the δ-neighborhood of the boundary ∂U_0 of the set U_0. Let $U_{1,\delta} = U_0 \backslash W_\delta(\partial U_0)$ and $U_{2,\delta} = U_0 \cup W_\delta(\partial U_0)$, thus the set $U_{1,\delta}$ tends to U_0 from inside as $\delta \to 0$, while $U_{2,\delta}$ tends to U_0 from outside as $\delta \to 0$. Since the limit cycle x_0 is isolated then there exists $\delta_0 > 0$ such that

$$G_0(x) \neq x \quad \text{for any} \quad x \in \partial W_{U_{1,\delta}} \cup \partial W_{U_{2,\delta}} \quad \text{and any} \quad \delta \in (0, \delta_0]. \tag{66}$$

Moreover, being $T > 0$, we can choose $\delta_0 > 0$ in such a way that

$$\psi(\xi) \neq 0 \quad \text{for any} \quad \xi \in \partial U_{1,\delta} \cup \partial U_{2,\delta} \quad \text{and any} \quad \delta \in [0, \delta_0]. \tag{67}$$

From (67) we get

$$d_B(\psi, U_{1,\delta_0}) = d_B(\psi, U_{2,\delta_0}) = d_B(\psi, U_0).$$

Since U_0 is the interior of the limit cycle x_0 of system (3) by Poincaré theorem, see Lefschetz ([21], Theorem 11.1) or Krasnosel'skii et al. ([19], Theorem 2.3) we have $d_B(\psi, U_0) = 1$ and so

$$d_B(\psi, U_{1,\delta_0}) = d_B(\psi, U_{2,\delta_0}) = 1.$$

In virtue of (66) and the fact that $W_U \cap \mathbb{R}^2 = U$, ([5], Corollary 1) applies to conclude that

$$d(I - G_0, W_{U_{1,\delta_0}}) = d_B(\psi, U_{1,\delta_0}) \quad \text{and} \quad d(I - G_0, W_{U_{2,\delta_0}}) = d_B(\psi, U_{2,\delta_0}),$$

hence

$$d(I - G_0, W_{U_{1,\delta_0}}) = 1 \quad \text{and} \quad d(I - G_0, W_{U_{2,\delta_0}}) = 1.$$

Therefore, there exists $\varepsilon_0 > 0$ such that

$$d(I - G_\varepsilon, W_{U_{1,\delta_0}}) = d(I - G_\varepsilon, W_{U_{2,\delta_0}}) = 1 \quad \text{for any } \varepsilon \in (0, \varepsilon_0). \tag{68}$$

Since by the definition of z_1 we have that $z_1(t+T) = \rho_* z_1(t)$ for any $t \in [0, T]$, then for any $t \in [0, T]$ it is easily seen that $f_1(\theta, t+T) = \rho_* f_1(\theta, t)$, whenever $\theta \in [0, T]$, and thus from (63) we have also that $f_1(\theta_0, t + \theta_0) \neq 0$ for any $t \in [0, T]$. Therefore, all the conditions of Theorem 2 are satisfied and we can take $\varepsilon_0 > 0$ sufficiently small to have

$$d(I - G_\varepsilon, W_{U_0}) = d_B(F, U_0) \quad \text{for any } \varepsilon \in (0, \varepsilon_0). \tag{69}$$

By (64), (68) and (69) we conclude that for any $\varepsilon \in (0, \varepsilon_0)$ there exist

$$x_{1,\varepsilon} \in W_{U_0} \backslash W_{U_{1,\delta_0}}, \quad \text{and} \quad x_{2,\varepsilon} \in W_{U_{2,\delta_0}} \backslash W_{U_0} \tag{70}$$

such that $G_\varepsilon(x_{1,\varepsilon}) = x_{1,\varepsilon}$ and $G_\varepsilon(x_{2,\varepsilon}) = x_{2,\varepsilon}$. From (70) we have that for any $\varepsilon \in (0, \varepsilon_0)$ there exist points $t_{1,\varepsilon}, t_{2,\varepsilon} \in [0, T]$ such that $x_{1,\varepsilon}(t_{1,\varepsilon}) \in U_0 \backslash U_{1,\delta_0}$ and $x_{2,\varepsilon}(t_{2,\varepsilon}) \in U_{2,\delta_0} \backslash U_0$. Thus $x_{1,\varepsilon}(t) \to \partial U_0$ and $x_{2,\varepsilon}(t) \to \partial U_0$, for any $t \in [0, T]$, as $\varepsilon \to 0$, otherwise there would exist a T-periodic solution x_* to system (3) and a point $t_* \in [0, T]$ such that either $x_*(t_*) \in U_0 \backslash U_{1,\delta_0}$ or $x_*(t_*) \in U_{2,\delta_0} \backslash U_0$ contradicting (66). Therefore, see ([25],

Theorem p. 287) or ([23], Lemma 2), for every $i \in \{1,2\}$ there exists $\theta_i \in [0,T]$ satisfying (65). The fact that $x_{1,\varepsilon}(t) \in U_0$ and $x_{2,\varepsilon}(t) \notin U_0$ for any $t \in [0,T]$ and $\varepsilon > 0$ sufficiently small follows from Corollary 2 and so the proof is complete.

□

Remark 4. *From the proof of Theorem 3 it results that* $d(I - G_\varepsilon, W_{U_0} \backslash \overline{W}_{U_1,\delta_0})$ *and* $d(I - G_\varepsilon, \overline{W}_{U_2,\delta_0} \backslash W_{U_0})$ *are different from zero for* $\varepsilon \in (0, \varepsilon_0)$*. This fact can be used to obtain stability properties of solutions* $x_{1,\varepsilon}$ *and* $x_{2,\varepsilon}$ *in the case when further information on the number of* T*-periodic solutions to (1) belonging to the sets* $W_{U_0} \backslash \overline{W}_{U_1,\delta_0}$ *and* $\overline{W}_{U_2,\delta_0} \backslash W_{U_0})$ *are available, see Ortega [31].*

3. An example.

In this section we always assume that condition (A_0) is satisfied. The well known formula by Poincaré, see Krasnoselskii et. al. ([19], formula 1.16) states that

$$\mathrm{ind}(x_0, F) = \frac{1}{2\pi} \int_0^T \frac{[\alpha(\theta)]_1 [\alpha'(\theta)]_2 - [\alpha(\theta)]_2 [\alpha'(\theta)]_1}{[\alpha^2(\theta)]_1 + [\alpha^2(\theta)]_2} d\theta,$$

where $\alpha(\theta) = F(x_0(\theta))$, $\theta \in [0,T]$. The relationship between $\mathrm{ind}(x_0, F)$ and $d_B(F, U_0)$ was discussed in Remark 2. In this section we show how the representation $F(x_0(\theta)) = f_0(\theta)\dot{x}_0(\theta) + f_1(\theta,\theta)y_1(\theta)$, $\theta \in [0,T]$, of the function F on ∂U_0 permits a simpler calculation of $d_B(F, U_0)$. For this, we consider the case when $\phi(t,\xi) = -\phi(t + T/2, \xi)$, which includes, in particular, the classes of perturbations $\phi(t,x) = \sin t \cdot \phi_1(x)$ and $\phi(t,x) = \cos t \cdot \phi_1(x)$, where $\phi_1 \in C(\mathbb{R}^2, \mathbb{R}^2)$. We can prove the following result.

Proposition 1. *Let* $\widetilde{F} \in C(\mathbb{R}^2, \mathbb{R}^2)$ *be a vector field such that* $\widetilde{F}(x_0(\theta)) = f_0(\theta)\dot{x}_0(\theta) + f_1(\theta, T)\dot{x}_0^\perp(\theta)$, $\theta \in [0,T]$. *Assume that*

$$\langle \dot{x}_0(0), z_0(0) \rangle = \langle y_1(0), z_1(0) \rangle = 1, \tag{71}$$

$$f_0(\theta) = -f_0(\theta + T/2) \quad for\ any\ \theta \in [0,T], \tag{72}$$

$$f_1(\theta, T) = -f_1(\theta + T/2, T) \quad for\ any\ \theta \in [0,T]. \tag{73}$$

Moreover, assume that there exists an unique $\theta_0 \in [0, T/2)$ *such that* $f_0(\theta_0) = 0$*. Finally, assume that the function* f_0 *is strictly monotone at the point* θ_0 *and that*

$$f_1(\theta_0, T) \neq 0. \tag{74}$$

Then either $d_B(\widetilde{F}, U_0) = 0$ *or* $d_B(\widetilde{F}, U_0) = 2$.

The proof of the proposition is based on the following technical lemma.

Lemma 5. *Let* $U \subset \mathbb{R}^2$ *be an open set whose boundary* ∂U *is a Jordan curve* $q : [0,T] \to \mathbb{R}^2$, *with* $q(0) = q(T)$*. Let* $\widetilde{F} : \mathbb{R}^2 \to \mathbb{R}^2$ *be a continuous vector field such that* $\widetilde{F}(\xi) \neq 0$ *for every* $\xi \in \partial U$*. Assume that for a continuous function* $z : [0,T] \to \mathbb{R}^2$, $z(0) = z(T)$*, the following conditions hold:*

1) $\langle z(\theta), \dot{q}(\theta) \rangle \neq 0$ *for every* $\theta \in [0,T]$,

2) the function $f(\theta) = \left\langle z(\theta), \widetilde{F}(q(\theta)) \right\rangle$ *has exactly two zeros* $\theta_1, \theta_2 \in [0,T)$,

3) the function f *is strictly monotone at* θ_1 *and* θ_2,

4) $\mathrm{sign} \left\langle z(\theta_1)^\perp, \widetilde{F}(q(\theta_1)) \right\rangle = -\mathrm{sign} \left\langle z(\theta_2)^\perp, \widetilde{F}(q(\theta_2)) \right\rangle$.

Then either $d_B(\widetilde{F}, U) = 0$ *or* $d_B(\widetilde{F}, U) = 2$.

Proof. Assume that the parametrization q is positive, namely the set U is on the left side if one follows ∂U according to the orientation given by q when t increases from 0 to T, otherwise we consider the opposite parametrization $\tilde{q}(\theta) = q(-\theta)$. For any $t \in [0,T]$ we denote by $\Theta(t)$ the angle (in radians) between the vectors $\dot{q}(0)$ and $\dot{q}(t)$ calculated in the counter-clockwise direction. Clearly $\Theta(t)$ is a multi-valued function of t. Let $\Gamma_{\dot{q}}(t)$ be the single-valued branch of $\Theta(t)$ such that $\Gamma_{\dot{q}}(0) = 0$ and let $Q : \partial U \to \mathbb{R}^2$ be the vector field defined by $Q(q(t)) := \dot{q}(t)$, whenever $t \in [0,T]$, hence $\Gamma_{\dot{q}}(t) = \Gamma_{Q\circ q}(t)$. Following ([19], §1.2) the function $t \to \Gamma_{\dot{q}}(t)$ is called the angle function of the vector field Q on the curve q . Analogously, considering the angle between $\widetilde{F}(q(0))$ and $\widetilde{F}(q(t))$, we can define the angle function $\Gamma_{\widetilde{F}\circ q}(t)$ of the vector field $\widetilde{F}$ on the curve q. By the definition of the rotation number for planar vector fields on the boundary of simply-connected sets, see ([19], § 1.3, formula 1.11) we have

$$d_B(\widetilde{F},U) = \frac{1}{2\pi}[\Gamma_{\widetilde{F}\circ q}(T) - \Gamma_{\widetilde{F}\circ q}(0)]. \tag{75}$$

Therefore, in order to prove the lemma we must calculate the right hand side of (75). For this, denote by $\widehat{h_1,h_2} \in [0,2\pi)$ the angle between the vectors h_1 and h_2 in the counter-clockwise direction, that is $\widehat{h_1,h_2} + \widehat{h_2,h_1} = 2\pi$. Observe that

$$\Gamma_{\widetilde{F}\circ q}(\theta) - \Gamma_{\dot{q}}(\theta) = \Gamma_{\dot{q},\widetilde{F}\circ q}(\theta) - \widehat{\dot{q}(0),\widetilde{F}(q(0))}, \tag{76}$$

where $\Gamma_{\dot{q},\widetilde{F}\circ q}(\theta)$ is the single valued branch of the multi-valued angle between $\dot{q}(\theta)$ and $\widetilde{F}(q(\theta))$ such that $\Gamma_{\dot{q},\widetilde{F}\circ q}(0) = \widehat{\dot{q}(0),\widetilde{F}(q(0))}$.

To calculate $\Gamma_{\dot{q},\widetilde{F}\circ q}(\theta)$ we introduce the function $\angle : \mathbb{R}^2 \times \mathbb{R}^2 \to [-\pi,\pi]$ as follows

$$\angle(h_1,h_2) = \begin{cases} \widehat{h_1,h_2} & \text{as } \widehat{h_1,h_2} \in [0,\pi], \\ \widehat{h_1,h_2} - 2\pi & \text{as } \widehat{h_1,h_2} \in (\pi,2\pi] \end{cases}$$

By condition 3) we have that $\mathrm{ind}(\theta_i,f) = +1$ or $\mathrm{ind}(\theta_i,f) = -1$ according to whether f is increasing or decreasing at θ_i, $i = 1,2$.

Up to a shift in time, since $\theta_2 - \theta_1 < T$, we may assume that the zeros θ_1, θ_2 of $f(\theta) = \left\langle z(\theta), \widetilde{F}(q(\theta))\right\rangle$ belong to the interval $(0,T)$.

Assume that $\langle z(\theta), \dot{q}(\theta)\rangle > 0$ for every $\theta \in [0,T]$, otherwise we consider $\tilde{z}(\theta) = z(-\theta)$ instead of $z(\theta)$. A possible way to write explicity the function $\Gamma_{\dot{q},\widetilde{F}\circ q}(\theta)$ is the following

$$\Gamma_{\dot{q},\widetilde{F}\circ q}(\theta) = \begin{cases} \angle(z(\theta),\dot{q}(\theta)) + \angle(\mathrm{sign}\left\langle z(\theta),\widetilde{F}(q(\theta))\right\rangle z(\theta),\widetilde{F}(q(\theta))) & \text{as } \theta \in [0,\theta_1), \\ \angle(z(\theta),\dot{q}(\theta)) + \angle(\mathrm{sign}\left\langle z(\theta),\widetilde{F}(q(\theta))\right\rangle z(\theta),\widetilde{F}(q(\theta))) + & \\ \qquad +\pi\,\mathrm{ind}(\theta_1,f)\mathrm{sign}\left\langle z(\theta_1)^\perp,\widetilde{F}(q(\theta_1))\right\rangle & \text{as } \theta \in (\theta_1,\theta_2), \\ \angle(z(\theta),\dot{q}(\theta)) + \angle(\mathrm{sign}\left\langle z(\theta),\widetilde{F}(q(\theta))\right\rangle z(\theta),\widetilde{F}(q(\theta))) + & \\ \qquad +\pi\,\mathrm{ind}(\theta_1,f)\mathrm{sign}\left\langle z(\theta_1)^\perp,\widetilde{F}(q(\theta_1))\right\rangle + & \\ \qquad +\pi\,\mathrm{ind}(\theta_2,f)\mathrm{sign}\left\langle z(\theta_2)^\perp,\widetilde{F}(q(\theta_2))\right\rangle & \text{as } \theta \in (\theta_2,T]. \end{cases}$$

It is easy to see that the above representation of the function $\theta \to \Gamma_{\dot{q},\widetilde{F}\circ q}(\theta)$ can be extend to θ_1 and θ_2 by continuity. Since

$$\begin{aligned} \theta &\to \angle(z(\theta),\dot{q}(\theta)), \\ \theta &\to \angle(\mathrm{sign}\left\langle z(\theta),\widetilde{F}(q(\theta))\right\rangle z(\theta),\widetilde{F}(q(\theta))) \end{aligned}$$

are T-periodic functions from (75)-(76), taking into account that

$$d_B(Q,U) = \frac{1}{2\pi}[\Gamma_{\dot q}(T) - \Gamma_{\dot q}(0)] = 1, \tag{77}$$

(see e.g. [19], Theorem 2.4), we have

$$d_B(\widetilde F,U) = 1 + \frac{1}{2}\left[\mathrm{ind}(\theta_1,f)\mathrm{sign}\left\langle z(\theta_1)^\perp, \widetilde F(q(\theta_1))\right\rangle + \mathrm{ind}(\theta_2,f)\mathrm{sign}\left\langle z(\theta_2)^\perp, \widetilde F(q(\theta_2))\right\rangle\right]. \tag{78}$$

Since the function f is T-periodic then

$$\mathrm{ind}(\theta_1,f) = -\mathrm{ind}(\theta_2,f) \tag{79}$$

By assumption 4) and (79) the claim can be easily derived from (78).

□

Proof of Proposition 1.
Let $U = U_0$, $q(t) = x_0(t)$, $z(t) = \dfrac{\dot x_0(t)}{\|\dot x_0(t)\|^2}$, $t \in [0,T]$, thus the function f_0 turns out to be the function f defined in Lemma 5. Let us now show that all the conditions of Lemma 5 hold. In fact, we have that $\langle \dot x_0(t), z(t)\rangle = 1$ for any $t \in [0,T]$ and so condition 1) is satisfied. Our assumptions imply that the function f_0 has only two zeros $\theta_1 = \theta_0$ and $\theta_2 = \theta_0 + T/2$ in the interval $[0,T]$ and it is strictly monotone at these points, thus conditions 2) and 3) of Lemma 5 are also satisfied. Finally, $\left\langle z(\theta)^\perp, \widetilde F(x_0(\theta))\right\rangle = f_1(\theta,T)$ and so (73) implies condition 4) of Lemma 5. Hence the proof is complete.

□

By combining Theorem 3 and Proposition 1 we obtain the following result.

Corollary 3. *Assume conditions (10) and assume that*

$$\phi(t,\xi) = -\phi(t+T/2,\xi) \quad \textit{for any } t\in[0,T] \;\; \textit{and any } \xi \in \mathbb{R}^2.$$

Moreover, assume that there exists a unique $\theta_0 \in [0,T/2)$ such that $f_0(\theta_0) = 0$. Finally, assume that the function f_0 is strictly monotone at the point θ_0 and

$$f_1(\theta_0,t) \neq 0 \quad \textit{for any } t \in [0,T]. \tag{80}$$

Then there exists $\varepsilon_0 > 0$ such that for every $\varepsilon \in (0,\varepsilon_0)$ system (1) has at least two T-periodic solutions $x_{1,\varepsilon}$ and $x_{2,\varepsilon}$ satisfying

$$x_{i,\varepsilon}(t-\theta_i) \to x_0(t) \quad \textit{as } \varepsilon \to 0,\; i = 1,2,$$

where $\theta_1,\theta_2 \in \{\theta_0, \theta_0 + T/2\}$. Furthermore, we have that $x_{1,\varepsilon}(t) \in U_0$ and $x_{2,\varepsilon}(t) \notin U_0$, for every $t\in[0,T]$ and $\varepsilon \in (0,\varepsilon_0)$.

Proof of Corollary 3.
To apply Theorem 3 we only have to verify condition (64). For this we will make use of Proposition 1. Without loss of generality we can assume that

$$\left\langle y_1(0), \dot x^\perp(0)\right\rangle > 0. \tag{81}$$

We claim that, under the conditions of Corollary 3, the vector field $\widetilde F$ of Proposition 1 is homotopic on ∂U_0 to the vector field F of Theorem 3. To prove the claim we show that the following homotopy joining $\widetilde F$ and F

$$D_\lambda(x_0(\theta)) = f_0(\theta)\dot x_0(\theta) + f_1(\theta, \lambda T + (1-\lambda)\theta)(\lambda \dot x_0(\theta)^\perp + (1-\lambda)y_1(\theta)), \quad \text{with } \theta\in[0,T] \quad \text{and } \lambda \in [0,1],$$

is admissible. Assume the contrary, therefore there exist $\lambda_0 \in [0,1]$ and $\theta_0 \in [0,T]$ such that

$$f_0(\theta_0)\dot{x}_0(\theta_0) + f_1(\theta_0, \lambda_0 T + (1-\lambda_0)\theta_0)(\lambda_0 \dot{x}_0(\theta_0)^\perp + (1-\lambda_0)y_1(\theta_0)) = 0.$$

By condition (81) we have that the vectors $\dot{x}_0(\theta_0)$ and $\lambda_0\dot{x}_0(\theta_0)^\perp + (1-\lambda_0)y_1(\theta_0)$ are linearly independent thus

$$f_0(\theta_0) = 0 \quad \text{and} \quad f_1(\theta_0, \lambda_0 T + (1-\lambda_0)\theta_0) = 0$$

contradicting assumption (80). Hence we have proved that

$$d_B(F, U_0) = d_B(\widetilde{F}, U_0).$$

Applying Proposition 1 we obtain that

$$d_B(F, U_0) \in \{0, 2\},$$

namely assumption (64) of Theorem 3 is satisfied and the conclusion of the corollary follows from Theorem 3.

□

At the end of the paper we would like to stress that all the functions y_1, z_0 and z_1 can be easily determined both analytically and numerically once the limit cycle x_0 is known. We give in the following a sketch of both approaches.

1) The analytical approach.

Since $\dot{x}_0$ is one of the two eigenfunctions of system (4) then by using well known formulas, see e.g. Pontrjagin ([33], p. 138), the dimension of the system (4) can be decreased by 1, thus the obtained one-dimensional system can be easily solved to determine y_1. Furthermore, by Lemma 1 the eigenfunctions z_0 and z_1 can be determined by the formula

$$(z_0(t)\ z_1(t)) = ((\dot{x}_0(t)\ y_1(t))^*)^{-1}.$$

2) A direct numerical approach.

From Lemma 1 we have $\langle \dot{x}_0(0), z_1(0)\rangle = 0$, therefore as initial condition we may take $z_1(0) = \dot{x}_0(0)^\perp$ and then z_1 can be obtained by a numerical computation. By the definition of z_1 there exists a T-periodic function $a \in C(\mathbb{R}, \mathbb{R}^2)$ such that $z_1(t) = a(t)\,\mathrm{e}^{\rho_* t}$. Assume, that $\rho_* < 0$. Let us fix an arbitrary vector $\xi \in \mathbb{R}^2$, which is linearly dependent with $z_1(0)$ and calculate the solution z of system (6) satisfying $z(0) = \xi$ on the interval $[0, kT]$ where $k \in \mathbb{N}$. It turns out that larger is k better accuracy is obtained. Observe that z can be represent by

$$z(t) = \alpha a(t)\mathrm{e}^{\rho_* t} + z_0(t), \tag{82}$$

where z_0 is an eigenfunction of (6), and since $\mathrm{e}^{\rho_* t} \to 0$ as $t \to +\infty$ then for given $k \in \mathbb{N}$ we may take

$$z_0(t) = z(t + (k-1)T) \quad \text{for any } t \in [0,T].$$

For the case $\rho_* > 0$ one should make the change of variables $\tilde{z}(t) = z(-t)$ for any $t \in \mathbb{R}$ in (6), to calculate $\tilde{z}_0$ on $[-kT, 0]$ (for this it is necessary to expand, the function z_1 on the interval $[-kT, 0]$) and then put $z_0(t) = z_1(-t)$ for any $t \in [0,T]$. Once z_0 is calculated with the desirable accuracy the function y_1 can be determined as the solution of (4) with initial condition $y_1(0) = z_0(0)^\perp$.

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